

BUILDING BLOCKS OF POLARIZED ENDOMORPHISMS OF NORMAL PROJECTIVE VARIETIES

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ABSTRACT. An endomorphism f of a projective variety X is polarized (resp. quasi-polarized) if $f^*H \sim qH$ for some ample (resp. nef and big) Cartier (integral) divisor H and integer $q > 1$. First, we use cone analysis to show that a quasi-polarized endomorphism is always polarized, and the polarized property descends via any equivariant dominant rational map. Next, we show that a suitable maximal rationally connected fibration (MRC) can be made f -equivariant using a construction of N. Nakayama, that f descends to a polarized endomorphism of the base Y of this MRC and that this Y is a Q -abelian variety (quasi-étale quotient of an abelian variety). Finally, we show that we can run the minimal model program (MMP) f -equivariantly for mildly singular X and reach either a Q -abelian variety or a Fano variety of Picard number one.

As a consequence, the building blocks of polarized endomorphisms are those of Q -abelian varieties and those of Fano varieties of Picard number one.

Along the way, we show that f always descends to a polarized endomorphism of the Albanese variety $\text{Alb}(X)$ of X , and that a power of f acts as a scalar on the Neron-Severi group of X (modulo torsion) when X is smooth and rationally connected.

Partial answers about X being of Calabi-Yau type, or Fano type are also given with an extra primitivity assumption on f which seems necessary by an example.

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1. INTRODUCTION

We work over an algebraically closed field k which has characteristic zero, and is uncountable (only used to guarantee the birational invariance of the rational connectedness property). Let f be a surjective endomorphism of an n -dimensional projective variety X . We say that f is *polarized* (resp. *quasi-polarized*) or the pair (X, f) is *polarized* (resp. *quasi-polarized*) by H , if there is an ample (resp. nef and big) \mathbb{Q} -Cartier \mathbb{Q} -divisor H such that $f^*H \sim_{\mathbb{Q}} qH$ (\mathbb{Q} -linear equivalence) for some integer $q > 1$. Equivalently, we may require $f^*H \sim qH$ (linear equivalence) after replacing H by a positive multiple. If X is a point, then the only trivial endomorphism is polarized by convention.

A surjective endomorphism of a projective variety is a finite morphism. In fact, such an endomorphism $f : X \rightarrow X$ induces an automorphism $f^* : N^1(X) \rightarrow N^1(X)$ of the real vector space $N^1(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for the Néron-Severi group $\text{NS}(X)$. So an ample divisor is the pull back of some divisor, which, together with the projection formula, imply the finiteness of f . Let $N_r(X)$ be the vector space of numerical equivalent classes of r -cycles; see definition in Section 2.1. We may define pullback of cycles for f , such that f^* induces an automorphism of $N_r(X)$ and $f_*f^* = (\deg f) \text{id}$; see [35, Section 2]. If X is normal, a Weil \mathbb{R} -divisor F is said to be *big* if $F = A + E$ for some ample \mathbb{Q} -Cartier divisor $A \in N^1(X)$ and pseudo-effective Weil \mathbb{R} -divisor E ; see Section 2.1.

If $f : X \rightarrow X$ is a surjective endomorphism such that $f^*H \sim_{\mathbb{Q}} qH$ for some nef and big divisor H and $q > 0$, then, by taking top self-intersection, the projection formula implies the relation between $\deg f$ and q : $\deg f = q^{\dim(X)}$.

Now we state our main results.

Proposition 1.1. *(cf. Proposition 3.7) Let $f : X \rightarrow X$ be a surjective endomorphism of an n -dimensional projective variety X and $q > 0$ a rational number. Assume one of the following two conditions.*

- (1) $f^*H \equiv qH$ (numerical equivalence) for some big \mathbb{R} -Cartier divisor H .
- (2) X is normal and $f^*H \equiv qH$ for some big Weil \mathbb{R} -divisor H .

*Then q is an integer and $f^*A \equiv qA$ for some ample Cartier (integral) divisor A . Further, if $q > 1$, then f is polarized. In particular, quasi-polarized endomorphisms are polarized.*

Given a normal projective variety X , denote by $\text{Aut}(X)$ the full automorphism group of X and $\text{Aut}_0(X)$ the neutral connected component of $\text{Aut}(X)$. Let B be a Weil \mathbb{R} -divisor. Denote by $\text{Aut}_{[B]}(X) := \{g \in \text{Aut}(X) \mid g^*B \equiv B\}$. If X is smooth and B is ample, then $[\text{Aut}_{[B]}(X) : \text{Aut}_0(X)] < \infty$ by [26, Proposition 2.2]. Generally, let G be a subgroup of $\text{Aut}(X)$, such that for any $g \in G$, $g^*B_g \equiv B_g$ for some big Cartier divisor

B_g . Then $[G : G \cap \text{Aut}_0(X)] < \infty$ by [9, Theorem 2.1]. Now applying Proposition 1.1, we can further weaken B_g as a big Weil \mathbb{R} -divisor in the following.

Theorem 1.2. (cf. Theorem 3.8) *Let X be a normal projective variety. Let G be a subgroup of $\text{Aut}(X)$, such that for any $g \in G$, $g^*B_g \equiv B_g$ for some big Weil \mathbb{R} -divisor B_g . Then $[G : G \cap \text{Aut}_0(X)] < \infty$.*

The polarized property descends via any equivariant dominant rational map. Indeed, we prove:

Theorem 1.3. (cf. Theorem 3.11) *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose f is polarized. Then g is polarized; and $(\deg f)^{\dim(Y)} = (\deg g)^{\dim(X)}$.*

Given a projective variety X , pick any smooth model $p : X' \rightarrow X$, we define the Albanese map alb_X of X as $\text{alb}_{X'} \circ p^{-1}$:

$$X \xrightarrow{p^{-1}} X' \xrightarrow{\text{alb}_{X'}} \text{Alb}(X') =: \text{Alb}(X).$$

Clearly, alb_X and $\text{Alb}(X)$ are independent of the choice of X' . By the universal property of the Albanese map, any surjective endomorphism (or even dominant rational self-map) f of X descends to a surjective endomorphism $f|_{\text{Alb}(X)}$ of $\text{Alb}(X)$. The following Corollary 1.4 is an application of Theorem 1.3.

Corollary 1.4. *Let X be a projective variety with a polarized endomorphism $f : X \rightarrow X$ and let $\text{alb}_X : X \dashrightarrow \text{Alb}(X)$ be the Albanese map of X . Then the following are true.*

- (1) alb_X is a dominant rational map.
- (2) The endomorphism $f|_{\text{Alb}(X)}$ of $\text{Alb}(X)$ induced from f is polarized.

Remark 1.5. Corollary 1.4 affirmatively answers Krieger - Reschke [25, Question 1.10].

We refer to [21, Chapters 2 and 5] for the definitions and the properties of log canonical (lc), Kawamata log terminal (klt), canonical and terminal singularities. A normal projective variety Y is *Q -abelian* if there exists a finite surjective morphism $A \rightarrow Y$ étale in codimension one (or *quasi-étale* in short) with A an abelian variety. By the ramification divisor formula, $K_Y \sim_{\mathbb{Q}} 0$.

A projective variety V of dimension n is *uniruled* if there is a dominant rational map $\mathbb{P}^1 \times U \dashrightarrow V$ with $\dim(U) = n - 1$. A normal projective variety V is *rationaly connected*, in the sense of Campana and Kollar-Miyaoka-Mori, ([6], [22]), if any two points of V are connected by a rational curve, which is equivalent to saying that two general points of V are connected by a rational curve since our ground field is uncountable (see [23]). Given

a uniruled normal projective variety X , there is a fibration: $\pi : X \dashrightarrow Y$, such that Y is a non-uniruled normal projective variety (cf. [15]), π is well defined over $\pi^{-1}(U)$ for an open dense subset $U \subseteq Y$ and the fibres of π over U are all rationally connected. We call it an *MRC* (maximal rationally connected) fibration in the sense of Campana and Kollar-Miyaoka-Mori and this fibration is unique up to birational equivalence (cf. [23]). The Albanese map of X always factors through the MRC fibration; see Lemma 4.2.

However, in general, fixing one MRC fibration, a surjective endomorphism of X descends only to a dominant rational self-map of Y . Nevertheless, we have the next result.

Proposition 1.6. *Let X be a normal projective variety with a polarized endomorphism $f : X \rightarrow X$. Then there is a special MRC fibration $\pi : X \dashrightarrow Y$ in the sense of Nakayama [30] (which is the identity map when X is non-uniruled) together with a (well-defined) surjective endomorphism g of Y , such that the following are true.*

- (1) $g \circ \pi = \pi \circ f$; g is polarized.
- (2) Y is Q -abelian (with only canonical singularities). Hence there is a finite Galois cover $T \rightarrow Y$ étale in codimension one with T an abelian variety and g lifts to a polarized endomorphism g_T of T .
- (3) Let $\bar{\Gamma}_{X/Y}$ be the normalization of the graph of π . Then the induced morphism $\bar{\Gamma}_{X/Y} \rightarrow Y$ is equi-dimensional with each fibre (irreducible) rationally connected.
- (4) If X has only klt singularities, then π is a morphism.

Remark 1.7. (1) By N. Fakhruddin (cf. [10]), the set of g -periodic points $\text{Per}(Y, g) := \{y \in Y \mid g^s(y) = y \text{ for some } s > 0\}$ is Zariski dense in Y . Thus the fibre $\bar{\Gamma}_y$ of the normalization of the graph of π over each $y \in \text{Per}(Y, g)$ is rationally connected and admits a polarized endomorphism. (2) By virtue of Proposition 1.6, the building blocks of polarized endomorphisms are those on Q -abelian varieties and those on rationally connected varieties.

In view of Remark 1.7, we still need to consider an arbitrary polarized endomorphism f of a rationally connected variety X . If X is mildly singular, we can run the minimal model program (MMP) and reach a Fano fibration provided that K_X is not pseudo-effective, and continue the MMP for the base of the Fano fibration and so on. Now the problem is that we need to guarantee the equivariance of the MMP with respect to f (or its positive power), i.e., to make sure that the extremal rays contracted during the MMP are fixed by f or its positive power. This is not easy, because there might be infinitely many extremal rays, being divisorial type, or flip type, or Fano type.

In Theorem 1.8, we manage to show the equivariance of the MMP with respect to a positive power of f , generalizing results in [35] for lower dimensions to all dimensions. This is done in Section 6.

For a surjective endomorphism $f : X \rightarrow X$ of a normal projective variety X , we say that a positive power of f^* is a *scalar* if $(f^s)^*$ induces a scalar action on $N^1(X)$ for some $s > 0$ (cf. Definition 7.1). Clearly, if a positive power of f^* is a scalar, then so is a positive power of $(f|_{\text{Alb}(X)})^*$. The converse may not hold even if we assume f to be polarized; see Example 7.3. Nevertheless, we have Theorem 1.8 (4) below.

Theorem 1.8. *Let (X, f) be a polarized pair such that X has at worst \mathbb{Q} -factorial klt singularities. Then, replacing f by a positive power, there exist a \mathbb{Q} -abelian variety Y , a morphism $X \rightarrow Y$, and an f -equivariant relative MMP over Y*

$$X = X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y$$

(i.e. $f = f_1$ descends to f_i on each X_i), with every $X_i \dashrightarrow X_{i+1}$ a divisorial contraction, a flip or a Fano contraction, of a K_{X_i} -negative extremal ray, such that we have:

- (1) If K_X is pseudo-effective, then $X = Y$ and it is \mathbb{Q} -abelian (see Proposition 1.6 or Lemma 4.7 for the lifting of f).
- (2) If K_X is not pseudo-effective, then for each i , $X_i \rightarrow Y$ is equi-dimensional holomorphic with every fibre (irreducible) rationally connected and f_i is polarized by some ample Cartier divisor H_i . The $X_{r-1} \rightarrow X_r = Y$ is a Fano contraction.
- (3) $N^1(X)$ is spanned by the pullbacks of $N^1(Y)$ and those $\{H_i\}_{i < r}$ which are f^* -eigenvectors corresponding to the same eigenvalue $q = (\deg f_i)^{1/\dim(X_i)}$ (independent of i).
- (4) f^* is a scalar: $f^*|_{N^1(X)} = q \text{ id}$, if and only if so is f_r^* .

Remark 1.9. In Theorem 1.8, if we weaken the klt singularities assumption on X to lc singularities, then our lemmas assert that we still can run f -equivariant MMP. We refer to [13] for the LMMP of \mathbb{Q} -factorial log canonical pair. However, we could not show that this MMP terminates and could not claim assertions (1) - (4) in Theorem 1.8.

A normal projective variety X is of *Calabi-Yau type* (resp. *Fano type*), if there is a boundary \mathbb{Q} -divisor $\Delta \geq 0$, such that the pair (X, Δ) has at worst lc (resp. klt) singularities and $K_X + \Delta \sim_{\mathbb{Q}} 0$ (resp. $-(K_X + \Delta)$ is ample). Applying Theorem 1.8 and working a bit more, we have the following result.

Theorem 1.10. *Let (X, f) be a polarized pair. Assume that X is smooth and rationally connected. Then, replacing f by a positive power, we have:*

- (1) f^* is a scalar. Namely, $f^*|_{N^1(X)} = q \text{ id}$ for some $q > 1$.

- (2) The number s of all prime divisors D_i with either $f^{-1}(D_i) = D_i$ or $\kappa(X, D_i) = 0$, satisfies $s \leq \dim(X) + \rho(X)$, where $\rho(X)$ is the Picard number of X .
- (3) The Iitaka D -dimensions satisfy $\kappa(X, -K_X) \geq \kappa(X, -(K_X + \sum_{i=1}^s D_i)) \geq 0$.
- (4) If (i) $s = \dim(X) + \rho(X)$, or (ii) $\kappa(X, -(K_X + \sum_{i=1}^s D_i)) = 0$ or (iii) D_1 is non-uniruled, then $K_X + \sum_{i=1}^s D_i \sim_{\mathbb{Q}} 0$ and hence X is of Calabi-Yau type (with $s = 1$ in Case (iii)).

Our Theorem 1.10 (3) is slightly stronger than that in [3, Theorem C], where they proved that $-K_X$ is pseudo-effective, but they did not assume X is rationally connected.

A polarized pair (X, f) is *imprimitive* if there is a dominant rational map $X \dashrightarrow Y$, with $0 < \dim(Y) < \dim(X)$, such that f descends to a polarized endomorphism f_Y of Y . A pair is *primitive* if it is not imprimitive.

Corollary 1.11. *Let (X, f) be a polarized pair such that:*

- (i) X has at worst \mathbb{Q} -factorial klt singularities,
- (ii) X is not a \mathbb{Q} -abelian variety, and
- (iii) (X, f^s) is primitive for all $s > 0$.

Then, replacing f by a positive power, we have the following assertions.

- (1) (X, f) is equivariantly birational to a Fano variety X_{r-1} of Picard number one, and f^* is a scalar. Precisely, running the MMP on X we get an f -equivariant sequence $X \dashrightarrow X_1 \cdots \dashrightarrow X_{r-1}$ of divisorial contractions and flips, such that X_{r-1} is a Fano variety of Picard number one with a polarized endomorphism f_{r-1} of X_{r-1} (induced from f).
- (2) Either X is of Calabi-Yau type, or $-K_X$ is big.

Remark 1.12. (1) Corollary 1.11 partially answers [38, Question 1.5] about X being of Calabi-Yau type. It also answers [38, Question 1.6 (2)] about X being of Fano type (but only up to f -equivariant birational map) with an extra primitivity assumption on the pair (X, f) which or something similar is needed, otherwise, Example 7.4 gives a negative answer in the general case.

(2) By Remark 1.7 and Corollary 1.11, we may say that the building blocks of polarized endomorphisms are those on (\mathbb{Q}) -abelian varieties and those on Fano varieties of Picard number one. This belief was stated and confirmed in [35] in dimension ≤ 4 .

The following question is natural from Theorem 1.10 without assuming X to be smooth.

Question 1.13. *Let (X, f) be a polarized pair. Assume that X is a rationally connected variety with at worst \mathbb{Q} -factorial terminal singularities. Is a positive power of f^* a scalar?*

The above question has a positive answer when $\dim(X) \leq 3$; see [35, Theorem 1.2]. Without the rational connectedness condition, Question 1.13 has a negative answer (see Example 7.2). In view of Example 7.3, though it is a $K3$ surface with canonical singularities, the terminality condition might be needed too.

2. PRELIMINARY RESULTS

2.1. Notation and terminology.

Let X be a projective variety of dimension n . We use Cartier divisor H (a Cartier divisor is integral, unless otherwise indicated) and its corresponding invertible sheaf $\mathcal{O}(H)$ interchangeably. Denote by $N^1(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for the Néron-Severi group $\text{NS}(X)$.

Given two r -cycles D_1 and D_2 , write $D_1 \equiv D_2$ if D_1 is *numerically equivalent* to D_2 , i.e. $D_1 \cdot H_1 \cdots H_r = D_2 \cdot H_1 \cdots H_r$ for any Cartier divisors H_1, \dots, H_r . Let $N_r(X)$ be the set of numerically equivalent classes of r -cycles with real coefficients. Elements in $N_{n-1}(X)$ are Weil \mathbb{R} -divisors. By Lemma 2.8 (i.e., [37, Lemma 3.2]), if X is normal, then $N^1(X)$ can be regarded as a subspace of $N_{n-1}(X)$ and they are the same if X is also \mathbb{Q} -factorial.

Denote by $\text{Amp}(X)$ the cone of all ample \mathbb{R} -Cartier divisors in $N^1(X)$, $\text{Nef}(X)$ the cone of all nef \mathbb{R} -Cartier divisors in $N^1(X)$, $\text{PEC}(X)$ the closure of the cone of all effective \mathbb{R} -Cartier divisors in $N^1(X)$, and $\text{PE}(X)$ the closure of the cone of all effective Weil \mathbb{R} -divisors in $N_{n-1}(X)$. Clearly, these cones contain no line. Let $f : X \rightarrow X$ be a finite surjective endomorphism. We may define pullback of cycles for finite surjective morphism, such that f^* induces an automorphism of $N_r(X)$ and $f_* f^* = (\deg f) \text{id}$; see [35, Section 2]. Then these cones are $(f^*)^{\pm 1}$ -invariant.

A Weil \mathbb{R} -divisor D is *pseudo-effective* if its class $[D] \in \text{PE}(X)$. If X is normal, a Weil \mathbb{R} -divisor F is big if $F = A + E$ for some ample \mathbb{Q} -Cartier divisor $A \in N^1(X)$ and pseudo-effective Weil \mathbb{R} -divisor E , see [14, Theorem 3.5] for equivalent definitions.

Define:

- (1) $q(X) = h^1(X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X)$ (the irregularity);
- (2) $\tilde{q}(X) = q(\tilde{X})$ with \tilde{X} a smooth projective model of X ; and
- (3) $q^{\natural}(X) = \sup\{\tilde{q}(X') \mid X' \rightarrow X \text{ is finite surjective and étale in codimension one}\}.$

Definition 2.2. Let V be a finite dimensional real vector space and $S \subseteq V$ a subset.

Denote by S° the interior part of S and $\partial S = \bar{S} - S^\circ$. We define $\dim(S)$ as the dimension of the vector space spanned by S .

The convex generated by S is defined as

$$\left\{ \sum_{i \in I} a_i x_i \mid a_i \geq 0, x_i \in S, \sum_{i \in I} a_i = 1, |I| < \infty \right\}.$$

Suppose S is bounded, i.e., there exists some $N > 0$, such that $|s| < N$ for any $s \in S$. Then $|\sum_{i \in I} a_i x_i| \leq \sum_{i \in I} a_i \max_{i \in I} \{|x_i|\} = \max_{i \in I} \{|x_i|\} < N$. So the closure of the convex generated by S is bounded. If S is a finite set, then the convex generated by S is a polytope which is covered by finitely many $\dim(S)$ -simplexes. So for arbitrary S of dimension m and the convex D generated by S , we may write $d = \sum_{i=1}^{m+1} a_i x_i$ with $a_i \geq 0$, $x_i \in S$ and $\sum_{i=1}^{m+1} a_i = 1$.

The cone generated by S is defined as

$$\{\sum_{i \in I} a_i x_i \mid a_i \geq 0, x_i \in S, |I| < \infty\}.$$

Let A and B be two subsets of V . We define $d(A, B) = \inf\{|a - b| : a \in A, b \in B\}$. Denote by $B(x, r) := \{x' \in V : |x - x'| < r\}$.

Let $C_1 \subseteq C_2$ be two cones of V . C_1 is called an *extremal face* of C_2 if $x, y \in C_2$ and $x + y \in C_1$ imply that $x, y \in C_1$. An extremal face of a closed cone is always closed.

Lemma 2.3. *Let C be a closed cone containing no lines and $C' \subseteq \partial C$ a subcone. Then C' is contained in a closed extremal face $F \subseteq \partial C$. In particular, C' is contained in a unique minimal closed extremal face in ∂C .*

Proof. Let $F := \{x \in C \mid x + y \in C' \text{ for some } y \in C\}$.

If $x_1, x_2 \in F$, then $x_1 + y_1 \in C'$ and $x_2 + y_2 \in C'$ for some $y_1, y_2 \in C$. For any $a \geq 0$, $ax_1 + ay_1 \in C'$ and $x_1 + x_2 + y_1 + y_2 \in C'$. So F is a cone.

Let V be the vector space spanned by C . If $x \in F \cap C^\circ$, then $x + y \in C'$ for some $y \in C$ and $B(x, r) \subseteq C^\circ$ for some $r > 0$. So $x + y \in B(x + y, r) = y + B(x, r) \subseteq C^\circ$ and hence $C' \cap C^\circ \neq \emptyset$, a contradiction. So $F \subseteq \partial C$.

If $x \in C'$, then $x + 0 = x \in C'$. So $C' \subseteq F$.

If $p, q \in C$ and $p + q \in F$, then $p + q + s \in C'$ for some $s \in C$. By the construction of F , we have both $p, q \in F$. So F is an extremal face.

Since C is closed, F is closed too.

By taking the intersection of all extremal faces containing C' , we get the minimal one. In fact, F is already the minimal one. Suppose F' is an extremal face containing C' . Then for any $x \in F$, $x + y \in C' \subseteq F'$ for some $y \in C$. So $x \in F'$. \square

Lemma 2.4. *Let C be a finite dimensional closed cone containing no lines and F a proper extremal face. Fix $d > 0$ and $k > 0$. Let S be the set of all $x \in C$ with $d(x, F) \geq d$ and $|x| \leq k$. Let B be the closure of the convex generated by S . Then $d(F, B) > 0$.*

Proof. Let B' be the convex generated by S . Set $n = \dim(B')$. Define $\eta(p, x_1, \dots, x_{n+1}) := d(p, D_{x_1, \dots, x_{n+1}})$, where $p \in F$, $x_i \in S$ and $D_{x_1, \dots, x_{n+1}}$ is the convex polytope generated by x_1, \dots, x_{n+1} . Clearly, η is a continuous function from $F \times S^{\times(n+1)}$ to $\mathbb{R}_{\geq 0}$. If $\eta(p, x_1, \dots, x_{n+1}) = 0$, then $p \in D_{x_1, \dots, x_{n+1}}$. Since F is an extremal face, $x_i \in F$, a contradiction. So $\eta > 0$. Let $F_{>2k} := \{p \in F : |p| > 2k\}$ and $F_{\leq 2k} = F - F_{>2k}$. Then $A := F_{\leq 2k} \times S^{\times(n+1)}$ is compact and hence $\eta|_A \geq d_1$ for some $d_1 > 0$. Note that $D(x_1, \dots, x_{n+1})$ is bounded by k . So for $A' := F_{>2k} \times S^{\times(n+1)}$, $\eta|_{A'} > k$. Since $B' = \bigcup_{x_1, \dots, x_{n+1} \in S} D_{x_1, \dots, x_{n+1}}$, $d(F, B') = \inf_{p \in F; x_1, \dots, x_{n+1} \in S} \{d(p, D_{x_1, \dots, x_{n+1}})\} \geq \min\{d_1, k\} > 0$ and hence $d(B, F) > 0$. \square

Proposition 2.5. *Let $f : V \rightarrow V$ be an automorphism of a positive dimensional real vector space V such that $f^{\pm 1}(C) = C$ for a closed cone $C \subseteq V$ which spans V and contains no line. Let q be a positive number. Then (1) and (2) below are equivalent.*

(1) $f(x) = qx$ for some $x \in C^\circ$.

(2) There exists a constant $N > 0$, such that $\frac{\|f^i\|}{q^i} < N$ for any $i \in \mathbb{Z}$.

Let $W \subseteq V$ be the eigenspace of f corresponding to the eigenvalue q . If (1) or (2) above is true, then f is a diagonalizable linear map with all eigenvalues of modulus q . So $F_\infty := \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{f^i}{q^i}$ is a well defined linear map onto W and $F_\infty|_W = \text{id}_W$.

Proof. We may assume $q = 1$ after replacing f by f/q .

(1) \Rightarrow (2) We may assume $|x| = 1$. Since $x \in C^\circ$ and C spans V , there exists some $t > 0$, such that $x + tv \in C$ for any $v \in V$ with $|v| \leq 1$. Since C contains no line, there exists some $s > 0$, such that $x - x' \notin C$ for any $x' \in C$ with $|x'| \geq s$. Suppose that $\|f^i\|$ is not bounded. Since C spans V , there exist some $n \in \mathbb{Z}$ and $y \in C$ with $|y| = 1$, such that $|tf^n(y)| > s$. Since $x - ty \in C$, $f^n(x - ty) \in C$. However, $f^n(x - ty) = x - tf^n(y) \notin C$, a contradiction.

Suppose (2) is true. If either the spectral radius of f is greater than 1 or f has a nontrivial Jordan block whose eigenvalue is of modulus 1. Then $\lim_{n \rightarrow +\infty} \|f^n\| = +\infty$, a contradiction. Similarly, the spectral radius of f^{-1} is 1. Therefore, the last assertion of the proposition follows.

(2) \Rightarrow (1) Set $n = \dim(V)$ and $m = \dim(W)$. If $m = n$, it is trivial. Suppose $m < n$. If $W \cap C^\circ \neq \emptyset$, then we are done. Suppose $W \cap C^\circ = \emptyset$. Since $f^{\pm 1}(C) = C$, $F_\infty(C) \subseteq C$ and hence $F_\infty(C) \subseteq W \cap C$. Note that W is spanned by $F_\infty(C)$. Then $W \cap C$ is an m -dimensional closed subcone of C in ∂C . By Lemma 2.3, $W \cap C \subseteq F$ for some minimal closed extremal face F of C in ∂C . Note that $f^{\pm 1}(F)$ are still minimal closed extremal faces in ∂C containing $W \cap C$. By the uniqueness, $f^{\pm 1}(F) = F$. Fix any $y \in C - F$ and $d := d(y, F) > 0$. Then by (2), for any $z \in F$, $d \leq |y - f^{-i}(z)| = |f^{-i}(f^i(y) - z)| <$

$N|f^i(y) - z|$. So $d(f^i(y), F) \geq \frac{d}{N}$ for any $i \in \mathbb{Z}$. Let B be the closure of the convex generated by all $f^i(y)$ ($i \in \mathbb{Z}$). By (2), B is bounded, $f^{\pm 1}(B) = B$ and $d(B, F) > 0$ by Lemma 2.4. Since $C \cap W \subseteq F$ and $B \subseteq C$, $B \cap W = B \cap C \cap W \subseteq B \cap F = \emptyset$. However, by Brouwer-fixed point theorem, $f(b) = b$ for some $b \in B$. So we get a contradiction. \square

Lemma 2.6. (cf. [21, Lemma 2.62]) *Let $f : X \rightarrow Y$ be a birational morphism. Assume that X is projective and Y is \mathbb{Q} -factorial. Then there is an effective f -exceptional divisor F such that $-F$ is f -ample.*

Proposition 2.7. (cf. [21, Proposition 1.45]) *Let $f : X \rightarrow Y$ be a morphism of projective varieties with M an ample divisor on Y . If L is an f -ample Cartier divisor on Y , then $L + \nu f^*M$ is ample for $\nu \gg 1$.*

Lemma 2.8. (cf. [37, Lemma 3.2]) *Let X be a projective variety of dimension n . Let H_1, \dots, H_{n-1} be ample \mathbb{R} -Cartier divisors and M an \mathbb{R} -Cartier divisor. Suppose that*

$$H_1 \cdots H_{n-1} \cdot M = 0 = H_1 \cdots H_{n-2} \cdot M^2.$$

Then $M \equiv 0$. In particular, if X is normal, then $N^1(X)$ is a subspace of $N_{n-1}(X)$.

The following lemma slightly extends [31, Lemma 2.2].

Lemma 2.9. *Let X be a projective variety of dimension n . Suppose that $D \in N^1(X)$ satisfies the following two conditions:*

- (1) $D \cdot G \cdot L_1 \cdots L_{n-2} \geq 0$ for any effective Cartier divisor G and any $L_i \in \text{Nef}(X)$.
- (2) $D \cdot H_1 \cdots H_{n-1} = 0$ for some nef and big \mathbb{R} -Cartier divisors H_1, \dots, H_{n-1} .

Then $D \equiv 0$.

Proof. By the proof of [31, Lemma 2.2], $D \cdot A^{n-1} = 0$ and $D^2 \cdot A^{n-2} = 0$ for some ample divisor A . So $D \equiv 0$ by Lemma 2.8. \square

Lemma 2.10. (cf. [35, Lemma 2.11]) *Let X be a normal projective variety and $f : X \rightarrow X$ a surjective endomorphism. Let $R_C := \mathbb{R}_{\geq 0}[C] \subseteq \overline{\text{NE}}(X)$ be an extremal ray (not necessarily K_X -negative). Then we have:*

- (1) $R_{f(C)}$ is an extremal ray.
- (2) If $f(C_1) = C$, then R_{C_1} is an extremal ray.
- (3) Denote by Σ_C the set of curves whose classes are in R_C . Then $f(\Sigma_C) = \Sigma_{f(C)}$.
- (4) If R_{C_1} is extremal, then $\Sigma_{C_1} = f^{-1}(\Sigma_{f(C_1)}) := \{D \mid f(D) \in \Sigma_{f(C_1)}\}$.

Let $f : X \rightarrow Y$ be a surjective morphism between normal projective varieties. Then f has connected fibres if and only if $f_*\mathcal{O}_X = \mathcal{O}_Y$ (cf. [18, Chapter III, Corollary 11.3 and Corollary 11.5]) and hence the composition of two such morphisms still has connected

fibres. In particular, the general fibre of f is connected if and only if all the fibres are connected and hence a birational morphism between normal projective varieties always has connected fibres (cf. [18, Chapter III, Corollary 11.4]). Suppose that f has connected fibres. Let X' be a resolution of X with $f' : X' \rightarrow Y$ the induced morphism. Then f' has connected fibres and the general fibre of f' is smooth by generic smoothness (cf. [18, Chapter III, Corollary 10.7]). In particular, the general fibre of f' is irreducible. Note that each fibre of f is an image of f' . So the general fibre of f is irreducible.

Now let $f : X \dashrightarrow Y$ be a dominant map between two normal projective varieties and $f_{\bar{\Gamma}} : \bar{\Gamma} \rightarrow Y$ the induced morphism with $\bar{\Gamma}$ the normalization of the graph of f . We say that f has *the general fibre rationally connected* if the general fibre of $f_{\bar{\Gamma}}$ is rationally connected. Let $f_1 : X_1 \rightarrow Y_1$ be a surjective morphism birationally equivalent to f with X_1 and Y_1 being normal projective. Then f has the general fibre rationally connected if and only if so does f_1 . This is because rational connectedness is a birational property (cf. [23, Chapter IV, Proposition 3.6]).

Lemma 2.11. *Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be two dominant maps between normal projective varieties. Suppose that f and g have the general fibre rationally connected. Then $g \circ f$ has the general fibre rationally connected.*

Proof. We may assume f and g are surjective morphisms between smooth projective varieties. Let U be an open dense subset of Y such that each fibre of f over U is rationally connected. Let V be an open dense subset of Z such that $g \circ f$ is smooth over V and each fibre of g over V is rationally connected and has a nonempty intersection with U . Then for any $z \in V$, we have a natural surjective morphism $f_z : X_z \rightarrow Y_z$, where $X_z = (g \circ f)^{-1}(z)$ while $Y_z = g^{-1}(z)$ is rationally connected. Since $Y_z \cap U \neq \emptyset$, f_z has the general fibre rationally connected. By [15, Corollary 1.3], X_z is rationally connected. So $g \circ f$ has the general fibre rationally connected. \square

Lemma 2.12. *Let X be a \mathbb{Q} -factorial klt normal projective variety. Suppose that $X = X_1 \dashrightarrow \cdots \dashrightarrow X_r = Y$ is a sequence of divisorial contractions, flips or Fano contractions, of K_{X_i} -negative extremal rays. Let $f : X \dashrightarrow Y$ be the composition. Then f has the general fibre rationally connected.*

Proof. By Lemma 2.11, it suffices to consider the case when f is a Fano contraction. Since X is klt, the general fibre of f is klt and hence rationally connected by [17, Corollary 1.3] and [17, Corollary 1.5]. \square

3. PROPERTIES OF (QUASI-) POLARIZED ENDOMORPHISMS

Lemma 3.1. *Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety X , such that $f^*H \equiv qH$ for some nef and big \mathbb{R} -Cartier divisor H and $q > 0$. Then $\deg f = q^{\dim(X)}$.*

Proof. Let $n = \dim(X)$. It is trivial if $n = 0$. Suppose that $n > 0$. By the projection formula, $(f^*H)^n = (\deg f)H^n = q^n H^n$. Since $H^n > 0$, $\deg f = q^n$. \square

Lemma 3.2. *Let $f : A \rightarrow A$ be a surjective endomorphism of an abelian variety A . Let Z be a subvariety of A such that $f^{-1}(Z) = Z$. Then $\deg f|_Z = \deg f$.*

Proof. f is étale by the ramification divisor formula and the purity of branch loci. Then $\deg f|_Z = |f^{-1}(z)| = \deg f$ for any $z \in Z$. \square

Lemma 3.3. *Let $f : X \rightarrow X$ be a polarized endomorphism of a projective variety X with $\deg f = q^{\dim(X)}$ and let Z be a closed subvariety of X with $f(Z) = Z$. Then $\deg f|_Z = q^{\dim(Z)}$.*

Proof. We may assume $f^*H \sim qH$ for some ample divisor H of X . Then $H|_Z$ is also an ample divisor of Z and $(f|_Z)^*(H|_Z) \sim q(H|_Z)$. So $\deg f|_Z = q^{\dim(Z)}$ by Lemma 3.1. \square

Lemma 3.4. *Let $f : X \rightarrow X$ be a quasi-polarized endomorphism with X normal projective and $\deg f = q^{\dim(X)} > 1$ (cf. Lemma 3.1). Then for any r -cycle $[D] \neq 0$ with $f^*D \equiv aD$, we have $|a| = q^{\dim(X)-r}$. Further, if D is effective, then $a = q^{\dim(X)-r}$.*

Proof. By [35, Lemma 2.2], every eigenvalue of $f^*|_{N^1(X)}$ is of modulus q . So $|a| = q^{\dim(X)-r}$ by [35, Lemma 2.4]. If D is effective, then $a > 0$ and hence $a = q^{\dim(X)-r}$. \square

The following lemma is true for not-necessarily normal projective varieties by using the same proof of [31, Lemma 2.3].

Lemma 3.5. (cf. [31, Lemma 2.3]) *Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety X such that $f^*H \equiv qH$ for some integer $q > 1$ and ample Cartier divisor H . Then there is an ample Cartier divisor $H' \equiv H$, such that $f^*H' \sim qH$. In particular, f is polarized.*

Lemma 3.6. *Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety X . Suppose that there is an ample \mathbb{R} -Cartier divisor H such that $f^*H \equiv qH$ for some rational number $q > 0$. Then q is an integer and $f^*H' \equiv qH'$ for some ample Cartier (integral) divisor H' .*

Proof. By Lemma 3.1, $q^{\dim(X)} = \deg f$. So q is an algebraic integer and also rational by assumption. Hence, q is an integer.

Let $W \subseteq N^1(X)$ be the eigenspace of $f^*|_{N^1(X)}$ with eigenvalue q and $W_{\mathbb{Q}}$ the set of all \mathbb{Q} -Cartier divisor classes in W . Note that $f^*|_{N^1(X)}$ is determined by $f^*|_{NS_{\mathbb{Q}}(X)}$. So $W_{\mathbb{Q}}$ is dense in W . Set $F_{\infty} := \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{(f^i)^*|_{N^1(X)}}{q^i}$. By Proposition 2.5, F_{∞} is a well defined projection from $N^1(X)$ onto W . Note that $N^1(X)$ is spanned by $\text{Amp}(X)$, $F_{\infty}(W) = W$, and F_{∞} is an open map from $N^1(X)$ onto W by Proposition 2.5. Then $F_{\infty}(H + \text{Amp}(X)) \cap W_{\mathbb{Q}} \neq \emptyset$. In particular, $H' := F_{\infty}(H + D)$ is an ample \mathbb{Q} -Cartier divisor for some $D \in \text{Amp}(X)$ and $f^*H' \equiv qH'$. Replacing H' by a multiple, we are done. \square

Proposition 3.7. *Let $f : X \rightarrow X$ be a surjective endomorphism of an n -dimensional projective variety X and $q > 0$ a rational number. Suppose one of the following is true.*

- (1) $f^*H \equiv qH$ for some big \mathbb{R} -Cartier divisor H .
- (2) X is normal and $f^*H \equiv qH$ for some big Weil \mathbb{R} -divisor H .

*Then q is an integer and $f^*A \equiv qA$ for some ample Cartier divisor A . Further, if $q > 1$, then f is polarized. In particular, quasi-polarized endomorphisms are polarized.*

Proof. Clearly, f^* induces automorphisms on $N_{n-1}(X)$ and $N^1(X)$.

For (1), applying Proposition 2.5 to $f^*|_{N^1(X)}$, $\text{PEC}(X)$ and its interior point H , Proposition 2.5(2) holds for $f^*|_{N^1(X)}$.

For (2), applying Proposition 2.5 to $f^*|_{N_{n-1}(X)}$, $\text{PE}(X)$ and its interior point H , Proposition 2.5(2) holds for $f^*|_{N_{n-1}(X)}$ and hence for $f^*|_{N^1(X)}$, since $N^1(X)$ is a subspace of $N_{n-1}(X)$ by Lemma 2.8.

Now for both (1) and (2), applying Proposition 2.5 to $f^*|_{N^1(X)}$ and $\text{Nef}(X)$, $f^*|_{N^1(X)}$ has an eigenvector in $\text{Nef}(X)^{\circ}$. So $f^*A \equiv qA$ for some ample \mathbb{R} -Cartier divisor A . By Lemma 3.6, q is an integer and we may assume A is Cartier.

If $q > 1$, then there exists an ample Cartier divisor $A' \equiv A$, such that $f^*A' \sim qA'$ by Lemma 3.5. \square

Let X be a projective variety. Denote by $\text{Aut}(X)$ the full automorphism group of X and $\text{Aut}_0(X)$ the neutral connected component of $\text{Aut}(X)$.

Theorem 3.8. *Let X be a normal projective variety. Let G be a subgroup of $\text{Aut}(X)$, such that for any $g \in G$, $g^*B_g \equiv B_g$ for some big Weil \mathbb{R} -divisor B_g . Then $[G : G \cap \text{Aut}_0(X)] < \infty$.*

Proof. Take an $\text{Aut}(X)$ -equivariant projective resolution $\pi : X' \rightarrow X$. We may regard G and $\text{Aut}_0(X)$ as subgroups of $\text{Aut}(X')$. By [4, Proposition 2.1], we can identify $\text{Aut}_0(X')$ with $\text{Aut}_0(X)$. For any $g \in G$, g fixes some ample class A_X in $\text{Amp}(X)$ by Proposition

3.7. Since π is birational, $A_{X'} = \pi^* A_X$ is big and g fixes $A_{X'}$. By [9, Theorem 2.1], $[G : G \cap \text{Aut}_0(X)] = [G : G \cap \text{Aut}_0(X')] < \infty$. \square

Remark 3.9. If π is a birational or a finite surjective morphism of projective varieties, then the pullback of a nef and big divisor is still nef and big. Therefore, this is also true when π is a generically finite surjective morphism by taking the Stein factorization. So the polarized property is preserved if the endomorphism is lifted under such morphisms by Proposition 3.7. By the universal property of normalization, if $f : X \rightarrow X$ is a polarized endomorphism of a projective variety X , then f lifts to a polarized endomorphism of the normalization of X . So we may assume X is normal in Corollary 1.4. In fact, we may further assume that $\text{alb}_X : X \dashrightarrow \text{Alb}(X)$ is a morphism, since f naturally lifts to a polarized endomorphism of the normalization of the graph of alb_X . Clearly, $\deg f$ is not changed during the replacements.

Lemma 3.10. *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two polarized endomorphisms such that $g \circ \pi = \pi \circ f$. Then the eigenvalues of $f^*|_{N^1(X)}$ are of modulus q if and only if so are the eigenvalues of $g^*|_{N^1(Y)}$ (if Y is a point, we assume this is always true). In particular, $(\deg f)^{\dim(Y)} = (\deg g)^{\dim(X)}$.*

Proof. By taking the graph of π and by Remark 3.9, we may assume π is a surjective morphism. Set $m = \dim(X)$ and $n = \dim(Y)$. Suppose that $f^*H_X = pH_X$ and $g^*H_Y \sim qH_Y$ for some ample divisors $H_X \in N^1(X)$, $H_Y \in N^1(Y)$ and $p, q > 1$. Since π is surjective, $\pi^* : N^1(Y) \rightarrow N^1(X)$ is an injection. If Y is not a point, then $N^1(Y)$ is of positive dimension. Applying Proposition 2.5 to the cones $\text{Nef}(X)$ and $\text{Nef}(Y)$, we have $p = q$. The last assertion then follows from Lemma 3.1. \square

Theorem 3.11. *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose f is polarized. Then g is polarized; and $(\deg f)^{\dim(Y)} = (\deg g)^{\dim(X)}$.*

Proof. By taking the graph of π and by Remark 3.9, we may assume π is a surjective morphism. Since f is polarized, $f^*|_{N^1(X)}$ satisfies Proposition 2.5(2). Since π is surjective, $N^1(Y)$ can be viewed as a subspace of $N^1(X)$ and hence $g^*|_{N^1(Y)}$ also satisfies Proposition 2.5(2). Therefore, Proposition 2.5(1) and Lemma 3.5 imply that g is polarized. The last formula follows from Lemma 3.10. \square

Corollary 3.12. *Let $\pi : X \dashrightarrow Y$ be a generically finite dominant rational map between two projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Then f is polarized if and only if so is g .*

Proof. By taking the graph of π and by Remark 3.9, we may assume π is a generically finite surjective morphism. If f is polarized, then g is polarized by Theorem 3.11. If g is polarized, then f is quasi-polarized and hence polarized, see Remark 3.9 and Proposition 3.7. \square

Lemma 3.13. *Let $f : X \rightarrow X$ be a polarized endomorphism of a normal projective variety X . Then either X is uniruled or $\kappa(X) = 0$.*

Proof. Suppose that X is non-uniruled. Then by [31, Theorem 3.2], X has only canonical singularities and $K_X \sim_{\mathbb{Q}} 0$. In particular, $\kappa(X) = 0$. \square

4. SPECIAL MRC FIBRATION AND THE NON-UNIRULED CASE

We refer to [30, Section 4] for the following result.

Lemma 4.1. (cf. [30, Theorem 4.19]) *Let $f : X \rightarrow X$ be a surjective endomorphism of a normal projective variety. Let $\pi : X \dashrightarrow Y$ be the special MRC fibration in the sense of [30, before Theorem 4.18] with Y non-uniruled (cf. [15]). Then there is an endomorphism $h : Y \rightarrow Y$ such that $\pi \circ f = h \circ \pi$.*

Lemma 4.2. *Let $\pi : X \dashrightarrow Y$ be an MRC fibration and $\text{alb}_X : X \dashrightarrow \text{Alb}(X)$ the Albanese map of X . Then there is a rational map $p : Y \dashrightarrow \text{Alb}(X)$, such that $p \circ \pi = \text{alb}_X$ and p is the Albanese map of Y .*

Proof. We may assume π and alb_X are morphisms. Note that π has the general fibre rationally connected and every map from a rationally connected variety to an abelian variety is trivial. So there is a Zariski dense subset U of Y such that $\text{alb}_X(\pi^{-1}(y))$ is a point for $y \in U$. By [20, Lemma 14], there is a rational map $p : Y \dashrightarrow \text{Alb}(X)$, such that $p \circ \pi = \text{alb}_X$. By the universal property of Albanese map, $p = \text{alb}_Y$ up to isomorphism. \square

Remark 4.3. By Remark 3.9, Lemmas 4.1, 4.2 and Theorem 3.11, it suffices to consider the non-uniruled normal projective X in Corollary 1.4. Let X be a normal projective variety with only canonical singularities and $K_X \sim_{\mathbb{Q}} 0$. Taking a resolution, we see the Kodaira dimension $\kappa(X) = 0$.

Lemma 4.4. *Let $f : X \rightarrow X$ be a polarized endomorphism of a non-uniruled normal projective variety X . Then X is Q -abelian with canonical singularities and $K_X \sim_{\mathbb{Q}} 0$. Further, there is an abelian variety A , such that the following diagram is commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\tau} & X \\ f_A \downarrow & & \downarrow f \\ A & \xrightarrow{\tau} & X \end{array}$$

where $f_A : A \rightarrow A$ is a polarized endomorphism and τ is a finite surjective morphism which is Galois and étale in codimension one.

Proof. By [16, Theorem 1.21] and [31, Theorem 3.2], X is Q -abelian with only canonical singularities.

By [31, Proposition 3.5], there exist an abelian variety A and a weak Calabi-Yau variety S , such that the following diagram is commutative:

$$\begin{array}{ccc} A \times S & \xrightarrow{\tau} & X \\ f_A \times f_S \downarrow & & \downarrow f \\ A \times S & \xrightarrow{\tau} & X \end{array}$$

where $f_A : A \rightarrow A$, $f_S : S \rightarrow S$ are polarized endomorphisms, and τ is a finite surjective morphism which is étale in codimension one. Since S is non-uniruled, S is Q -abelian by [16, Theorem 1.21] and hence $\dim(S) = q^\circ(S) = 0$ in the notation of [31] as in their definition of weak Calabi-Yau. Replacing A by the (unique) Albanese closure of X in codimension one and lifting f , we are done; see [31, Lemma 2.12]. \square

Remark 4.5. Assume X is a Q -abelian variety or just assume there is a finite surjective morphism $A \rightarrow X$ with A an abelian variety. Since A is a homogeneous variety, any effective divisor on A is nef. The same holds on X by the projection formula. Hence if $f : X \rightarrow X$ is quasi-polarized, it is also polarized without using Proposition 3.7.

We refer to the proof of [31, Proposition 3.5] for the following lemma.

Lemma 4.6. *Let X be a normal projective variety with klt singularities and $K_X \sim_{\mathbb{Q}} 0$ and let $\sigma : \hat{X} \rightarrow X$ be the global index-one cover. Then for any surjective endomorphism $f : X \rightarrow X$, there is a surjective endomorphism $\hat{f} : \hat{X} \rightarrow \hat{X}$ such that $\sigma \circ \hat{f} = f \circ \sigma$.*

Lemma 4.7. *Let X be a normal projective variety with klt singularities and $K_X \sim_{\mathbb{Q}} 0$ (this is the case when X is Q -abelian) and let $f : X \rightarrow X$ be a polarized endomorphism. Then there exist a finite surjective Galois cover $\tau : A \rightarrow X$ étale in codimension one with A an abelian variety and a polarized endomorphism $f_A : A \rightarrow A$, such that $\tau \circ f_A = f \circ \tau$. In particular, X is Q -abelian.*

Proof. Let $\sigma : \hat{X} \rightarrow X$ be the global index-one cover, i.e. the minimal cyclic covering satisfying $K_{\hat{X}} \sim 0$. Then there is a polarized endomorphism $\hat{f} : \hat{X} \rightarrow \hat{X}$ satisfying $\sigma \circ \hat{f} = f \circ \sigma$ by Lemmas 4.6 and Corollary 3.12. Since $K_{\hat{X}}$ is Cartier and \hat{X} is klt, \hat{X} has canonical singularities. In particular, $\kappa(\hat{X}) = 0$ by Remark 4.3 and hence \hat{X} is non-uniruled. Now by Lemma 4.4, there exist a finite surjective morphism $\tau' : A \rightarrow \hat{X}$ which is étale in codimension one with A an abelian variety and a polarized endomorphism

$f_A : A \rightarrow A$, such that $\tau' \circ f_A = \hat{f} \circ \tau'$. Now the lemma is proved since $\tau = \sigma \circ \tau'$ is still étale in codimension one by the ramification divisor formula, and by replacing A as in the proof of Lemma 4.4. \square

Lemma 4.8. *Let X be a Q -abelian variety and $f : X \rightarrow X$ a surjective endomorphism. Assume the existence of a non-empty closed subset $Z \subsetneq X$ and $s > 0$, such that $f^{-s}(Z) = Z$. Then f is not polarized.*

Proof. Replacing f by f^s , we may assume $f^{-1}(Z) = Z$. Suppose that f is polarized. By Lemma 4.7, there exist a finite surjective morphism $\tau : A \rightarrow X$ with A an abelian variety and a polarized endomorphism $f_A : A \rightarrow A$, such that $\tau \circ f_A = f \circ \tau$. Clearly, $f_A^{-1}(\tau^{-1}(Z)) = \tau^{-1}(Z)$. So we may assume that X is an abelian variety; and replacing f by a positive power, we may also assume that Z is irreducible. By Lemma 3.2, $\deg f|_Z = \deg f$; and by Lemma 3.3, $\deg f|_Z = (\deg f)^{\dim(Z)/\dim(X)}$. Since $\dim(Z) < \dim(X)$ and $\deg f > 1$ by Lemma 3.1, we get a contradiction. \square

5. PROOF OF COROLLARY 1.4 AND PROPOSITION 1.6

We begin with the following lemmas.

Lemma 5.1. *(cf. [32, Proposition 2.3] or [19, Lemma 8.1]) Let X be a normal projective variety having rational singularities (i.e. there exists a resolution $f : Y \rightarrow X$ such that $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$). Then $f^* : \text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$ is an isomorphism, and alb_X is a morphism. In particular, if $h^1(X, \mathcal{O}_X) \neq 0$, then alb_X is nontrivial.*

Lemma 5.2. *Let $\pi : X \rightarrow Y$ be a surjective morphism between normal projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two polarized endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose that Y is Q -abelian. Then the following are true.*

- (1) π is equi-dimensional.
- (2) If the general fibre of π is connected, then all the fibres of π are irreducible.
- (3) If the general fibre of π is rationally connected, then all the fibres of π are rationally connected.

Proof. Let

$$\Sigma_1 := \{y \in Y \mid \dim(\pi^{-1}(y)) > \dim(X) - \dim(Y)\},$$

$$\Sigma_2 := \{y \in Y \mid \pi^{-1}(y) \text{ is not irreducible}\},$$

and

$$\Sigma_3 := \{y \in Y \mid \pi^{-1}(y) \text{ is not rationally connected}\}.$$

Then $g^{-1}(\Sigma_i) = \Sigma_i$ for each i . Let $\overline{\Sigma_i}$ be the Zariski closure of Σ_i in Y . Then $g^{-1}(\overline{\Sigma_i}) = \overline{\Sigma_i}$. Note that $\overline{\Sigma_1} = \Sigma_1 \neq Y$; $\overline{\Sigma_2} \neq Y$ if π has the general fibre connected; and $\overline{\Sigma_3} \neq Y$ if π has the general fibre rationally connected. By Lemma 4.8, $\Sigma_i = \emptyset$ for each i . \square

Lemma 5.3. *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between normal projective varieties. Suppose that (X, Δ) is a klt pair for some effective \mathbb{Q} -divisor Δ and Y is \mathbb{Q} -abelian. Suppose further that the normalization of the graph $\Gamma_{X/Y}$ is equi-dimensional over Y (this holds when $f : X \rightarrow X$ is polarized and f descends to some polarized $f_Y : Y \rightarrow Y$; see Lemma 5.2). Then π is a morphism.*

Proof. Let W be the normalization of the graph $\Gamma_{X/Y}$ and $p_1 : W \rightarrow X$ and $p_2 : W \rightarrow Y$ the two projections. Let $\tau_1 : A \rightarrow Y$ be a finite surjective morphism étale in codimension one with A an abelian variety. Let W' be the normalization of $W \times_Y A$ and $\tau_2 : W' \rightarrow W$ and $p'_2 : W' \rightarrow A$ the two projections. Taking the Stein factorization of the composition $W' \rightarrow W \rightarrow X$, we get a birational morphism $p'_1 : W' \rightarrow X'$ and a finite morphism $\tau_3 : X' \rightarrow X$.

$$\begin{array}{ccccc} X' & \xleftarrow{p'_1} & W' & \xrightarrow{p'_2} & A \\ \downarrow \tau_3 & & \downarrow \tau_2 & & \downarrow \tau_1 \\ X & \xleftarrow{p_1} & W & \xrightarrow{p_2} & Y \end{array}$$

Since p_2 is equi-dimensional, by the base change, τ_2 is étale in codimension one. Let $U \subseteq X$ be the domain of $p_1^{-1} : X \dashrightarrow W$. Then, $\text{codim}(X - U) \geq 2$, and the restriction $\tau_3^{-1}(U) \rightarrow U$ of τ_3 is étale in codimension one, since so is τ_2 . Therefore, τ_3 is étale in codimension one. In particular, by the ramification divisor formula, $K_{X'} + \Delta' = \tau_3^*(K_X + \Delta)$ with $\Delta' = \tau_3^*\Delta$ an effective \mathbb{Q} -divisor. Since (X, Δ) is klt, (X', Δ') is klt by [21, Proposition 5.20] and hence X' has rational singularities by [21, Theorem 5.22]. Clearly, $\pi' := p'_2 \circ p_1'^{-1} : X' \dashrightarrow A$ is a dominant rational map, since p'_1 is birational and p'_2 is surjective. Then π' is a surjective morphism (with $p'_2 = \pi' \circ p'_1$) by Lemma 5.1 and the universal property of the Albanese map. Suppose π is not defined over some closed point $x \in X$. Then $\dim(W_x) > 0$ with $W_x = p_1^{-1}(x)$ and $\dim(p_2(W_x)) > 0$ by [8, Lemma 1.15]. Hence, $\dim(p'_2(\tau_2^{-1}(W_x))) > 0$ and then $\dim(p'_1(\tau_2^{-1}(W_x))) > 0$. However, $p'_1(\tau_2^{-1}(W_x)) = \tau_3^{-1}(x)$ has only finitely many points. This is a contradiction. \square

Lemma 5.4. *Let X be a projective variety with a polarized endomorphism $f : X \rightarrow X$. Then the Albanese map $\text{alb}_X : X \dashrightarrow \text{Alb}(X)$ is a dominant rational map.*

Proof. By Remark 3.9, Lemmas 4.1, 4.2 and Theorem 3.11, it suffices to consider the case when X is a non-uniruled normal projective variety. By Lemma 4.4, we can further assume X has only canonical singularities with $K_X \sim_{\mathbb{Q}} 0$. In particular, $\kappa(X) = 0$. By [20, Theorem 1] and Lemma 5.1, $\text{alb}_X : X \dashrightarrow \text{Alb}(X)$ is a surjective morphism. \square

Proof of Corollary 1.4. (1) follows from Lemma 5.4. (2) follows from (1) and Theorem 3.11. \square

Proof of Proposition 1.6. (1) follows from Lemmas 4.1 and Theorem 3.11; see [30, Corollary 4.20] for a different proof of g being polarized. (2) follows from Lemma 4.4. (3) follows from Lemma 5.2. (4) follows from Lemma 5.3. \square

6. MINIMAL MODEL PROGRAM FOR POLARIZED PAIRS

We follow the approach in [35, Lemma 2.10] and get the following general result:

Lemma 6.1. *Let (X, f) be a polarized pair. Suppose $A \subseteq X$ is a positive-dimensional subvariety with $f^{-i}f^i(A) = A$ for all $i \geq 0$. Then $M(A) := \{f^i(A) \mid i \geq 0\}$ is a finite set.*

Proof. Suppose that $\dim(X) = n$, and $f^*H \sim qH$ for some ample Cartier divisor H and $q > 1$. We shall prove by induction on the codimension of A in X . It is trivial if $A = X$.

Set $k := \dim(A) < \dim(X)$, $A_i := f^i(A)$ ($i \geq 0$).

Let Σ be the union of $\text{Sing}(X)$ and the irreducible components in the ramification divisor R_f of f .

We first claim that A_i is contained in Σ for infinitely many i . Otherwise, replacing A by some A_{i_0} , we may assume that A_i is not contained in Σ for all $i \geq 0$. So we have $f^*A_{i+1} = a_i A_i$ with $a_i \in \mathbb{Z}_{>0}$. So

$$q^n H^k \cdot A_{i+1} = (f_V^* H)^k \cdot f_V^* A_{i+1} = a_i q^k H^k \cdot A_i,$$

$$1 \leq H^k \cdot A_{i+1} = \frac{a_i}{q^{n-k}} \cdots \frac{a_1}{q^{n-k}} H^k \cdot A_1.$$

Thus $a_i \geq q^{n-k}$ for infinitely many i . So $A_i \subseteq \Sigma$ for infinitely many i , a contradiction. This proves the claim.

If $k = m - 1$, by the claim, $f^{i_1}(A) = f^{i_2}(A)$ for some $0 < i_1 < i_2$. Then $|M(A)| < i_2$.

If $k \leq m - 2$, assume that $|M(A)| = \infty$. Let B be the Zariski-closure of the union of those A_{i_1} contained in Σ . Then $k + 1 \leq \dim(B) \leq m - 1$, and $f^{-i}f^i(B) = B$ for all $i \geq 0$. Choose $r \geq 1$ such that $B' := f^r(B), f(B'), f^2(B'), \dots$ all have the same number of irreducible components. Let X_1 be an irreducible component of B' of maximal dimension. Then $k + 1 \leq \dim(X_1) \leq m - 1$ and $f^{-i}f^i(X_1) = X_1$ for all $i \geq 0$. By induction, $M(X_1)$ is a finite set. So we may assume that $f^{-1}(X_1) = X_1$, after replacing f by a positive power and X_1 by its image. Note that $f|_{X_1}$ is polarized. Now the codimension of A_{i_1} in X_1 is smaller than that of A in X . By induction, $M(A_{i_1})$ and hence $M(A)$ are finite. This is a contradiction. \square

Let X be a log canonical (lc) normal projective variety. We refer to [12, Theorem 1.1] for the cone theorem and [2, Corollary 1.2] for the existence of log canonical flips.

Lemma 6.2. *Let X be a \mathbb{Q} -factorial lc normal projective variety and $f : X \rightarrow X$ a surjective endomorphism. Let $\pi : X \rightarrow Y$ be a contraction of a K_X -negative extremal ray $R_C := \mathbb{R}_{\geq 0}[C]$. Suppose that $E \subseteq X$ is a subvariety such that $\dim(\pi(E)) < \dim(E)$ and $f^{-1}(E) = E$. Then replacing f by a positive power, $f(R_C) = R_C$; hence, π is f -equivariant.*

Proof. Since $\dim(\pi(E)) < \dim(E)$, we may assume $C \subseteq E$. By the cone theory, we have the linear exact sequence

$$0 \rightarrow N^1(Y) \xrightarrow{\pi^*} N^1(X) \xrightarrow{\cdot C} \mathbb{R} \rightarrow 0.$$

So $\pi^* N^1(Y)$ is a hyperplane in $N^1(X)$. Denote by $L = \pi^* N^1(Y)|_E$, which is also a hyperplane in $N^1(X)|_E$, since $H|_E \cdot C \neq 0$ for some ample divisor $H \in N^1(X)$. Let

$$S = \{D|_E \in N^1(X)|_E : (D|_E)^{\dim(E)} = 0\}.$$

Then $L \subseteq S$, since $\dim(\pi(E)) < \dim(E)$ and by the projection formula.

Since $(H|_E)^{\dim(E)} > 0$ for any ample divisor H in $N^1(X)$, S is a hypersurface in $N^1(X)|_E$. It is easy to see that f^* also induces an automorphism of $N^1(X)|_E$. Note that $f^*(E) \equiv aE$ for some $a > 0$, and $(f^*D)^{\dim(E)} \cdot E = \frac{\deg f}{a} D^{\dim(E)} \cdot E$. Hence, $D \in S$ if and only if $f^*D \in S$. This implies that S is f^* -invariant. L is an irreducible component of S . So replacing f by a positive power, $(f|_E)^*(L) = L$. By Lemma 2.10, $f^{-1}(R_C) = R_{C'}$ for some curve C' and $f_*(C') = eC$ for some $e > 0$. We may assume $C' \subseteq E$. For any $D \in N^1(Y)$, $\pi^*D|_E = (f|_E)^*(\pi^*D'|_E) = (f^*\pi^*D')|_E$ for some $D' \in N^1(Y)$. Then

$$\pi^*D \cdot C' = f^*\pi^*D' \cdot C' = f_*(f^*\pi^*D' \cdot C') = \pi^*D' \cdot eC = 0.$$

Thus, $R_{C'} = R_C$ and hence $f(R_C) = R_C$. The last assertion is true since the contraction π is uniquely determined by the ray R_C . \square

Remark 6.3. In Lemma 6.2, if $E = X$ i.e., if π is a Fano contraction, then π is f^s -equivariant for some $s > 0$. This is also a corollary of [34, Theorem 2.2] by showing that X has only finitely many Fano contractions.

Lemma 6.4. *Let (X, f) be a polarized pair with X being a \mathbb{Q} -factorial lc normal projective variety. Suppose that $\pi : X \rightarrow X_1$ is a divisorial contraction of a K_X -negative extremal ray $R_C := \mathbb{R}_{\geq 0}[C]$. Then f^s descends to a surjective endomorphism of X_1 for some $s > 0$.*

Proof. Let E be the exceptional divisor. Then E is irreducible (cf. [21, Proposition 2.5]). By Lemma 2.10, $f^{-i}f^i(E) = E$ for all $i \geq 0$. By Lemma 6.1, $M(E)$ is a finite set. So we may assume $f^{-1}(E) = E$ after replacing f by its positive power. Then π is f^s -equivariant for some $s > 0$ by Lemma 6.2. \square

Lemma 6.5. *Let (X, f) be a polarized pair with X being a \mathbb{Q} -factorial lc normal projective variety. Let $\sigma : X \dashrightarrow X^+$ be a flip with $\pi : X \rightarrow Y$ the corresponding flipping contraction of a K_X -negative extremal ray $R_C := \mathbb{R}_{\geq 0}[C]$. Then the commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad \sigma \quad} & X^+ \\ & \searrow \pi & \swarrow \pi^+ \\ & Y & \end{array}$$

is f^s -equivariant for some $s > 0$.

Proof. Let E be the exceptional locus of π . By Lemma 2.10, $f^{-i}f^i(E) = E$ for all $i \geq 0$. Choose $i_0 \geq 0$ such that $E' := f^{i_0}(E), f(E'), f^2(E'), \dots$ all have the same number of irreducible components. Then $f^{-i}f^i(E'(k)) = E'(k)$ for every irreducible component $E'(k)$ of E' . By Lemmas 6.1, $M(E'(k))$ is a finite set. Then $f^r(E'(k)) = f^s(E'(k))$ for some $r > s \geq 0$ and hence $f^{-i_0}(E'(k)) = f^{-i_0}(f^{-r}f^r(E'(k))) = f^{-i_0}(f^{-r}f^s(E'(k))) = f^{-(r-s)}(f^{-i_0}(E'(k)))$. So $f^{-(r-s)}$ permutes the irreducible components of $f^{-i_0}(E'(k))$. Let $E(k)$ be an irreducible component of E such that $f^{i_0}(E(k)) = E'(k)$. Then $f^{-t}(E(k)) = E(k)$ for some $t > 0$. Since $\dim(\pi(E(k))) < \dim(E(k))$, we have $f^s(R_C) = R_C$ for some $s > 0$ by applying Lemma 6.2 to f^t and $E(k)$. Hence, the rational maps on Y and X^+ induced from f^s are well defined morphisms by the following Lemma 6.6. \square

The following lemma is true by using the same proof of [35, Lemma 3.6] since log canonical flips exist.

Lemma 6.6. (cf. [35, Lemma 3.6]) *Let X be a \mathbb{Q} -factorial normal projective variety with at worst lc singularities, $f : X \rightarrow X$ a surjective endomorphism, and $X \dashrightarrow X^+$ a flip with $\pi : X \rightarrow Y$ the corresponding flipping contraction of a K_X -negative extremal ray $R_C := \mathbb{R}_{\geq 0}[C]$. Suppose that $R_{f(C)} = R_C$. Then the dominant rational map $f^+ : X^+ \dashrightarrow X^+$ induced from f , is holomorphic. Both f and f^+ descend to one and the same endomorphism of Y .*

Definition 6.7. (cf. [28]) Let X be a normal projective variety and D an \mathbb{R} -Cartier divisor. We say D is *movable* if: for any $\epsilon > 0$, any ample divisor H , and any prime divisor Γ , there is an effective \mathbb{R} -divisor Δ such that $\Delta \equiv D + \epsilon H$ and $\Gamma \not\subseteq \text{Supp } \Delta$.

Lemma 6.8. *Let X be a normal projective variety with only klt singularities and $f : X \rightarrow X$ a polarized endomorphism. Suppose that K_X is movable. Then X is \mathbb{Q} -abelian.*

Proof. First we claim that $K_X \sim_{\mathbb{Q}} 0$. Since K_X is movable, K_X is pseudo-effective and satisfies the first condition of Lemma 2.9. Suppose that $f^*H \sim_{\mathbb{Q}} qH$ for some ample

divisor H of X and $q > 1$. Taking intersection numbers with $(f^*H)^{n-1} = f^*H \cdots f^*H$ of the both sides of the ramification divisor formula $K_X = f^*K_X + R_f$, we obtain

$$(q-1)K_X \cdot H^{n-1} + R_f \cdot H^{n-1} = 0.$$

Since K_X and R_f are pseudo-effective, $K_X \cdot H^{n-1} = 0$. So by Lemma 2.9, $K_X \equiv 0$ and hence $K_X \sim_{\mathbb{Q}} 0$ by [28, Chapter V, Corollary 4.9].

Now the lemma follows from Lemma 4.7. \square

Lemma 6.9. *Let (X, f) be a polarized pair with X being a \mathbb{Q} -factorial klt normal projective variety. Assume that K_X is pseudo-effective. Then X is Q -abelian.*

Proof. We run MMP on X . Since K_X is pseudo-effective, we will never arrive at a non-birational contraction. By running finitely many steps, we get a birational map $\pi : X \dashrightarrow Y$ such that we will never meet any divisorial contraction starting from Y . Replacing f by a positive power, f descends step by step by Lemmas 6.4, 6.5, and Theorem 3.11. Let $g = f|_Y$ and $g^*H \sim_{\mathbb{Q}} qH$ for some ample Cartier divisor H of Y and $q > 1$.

We claim that K_Y is movable. Take an ample divisor A of Y and a sequence of positive numbers t_j approaching 0. By [1], we can run $(K_Y + t_j A)$ -MMP on Y with scaling of A to obtain a birational map

$$\pi_j : (Y, t_j A) \dashrightarrow (Y_j, t_j A_j),$$

such that $K_{Y_j} + t_j A_j$ is nef and hence movable. Since Y, Y_j are isomorphic in codimension 1 and they are all \mathbb{Q} -factorial, we have a natural isomorphism $\pi_j^* : N^1(Y_j) \rightarrow N^1(Y)$ and $D \in N^1(Y_j)$ is movable if and only if $\pi_j^* D$ is movable. So $K_Y + t_j A = \pi_j^*(K_{Y_j} + t_j A_j)$ is movable for each j . In particular, K_Y is movable.

Note that Y is Q -abelian by Lemma 6.8 and g is polarized. Then $X \rightarrow Y$ is a birational equi-dimensional morphism by Lemma 5.3. This is possible only when $X \rightarrow Y$ is an isomorphism by the Zariski's main theorem (cf. [18, Chapter V, Theorem 5.2]). \square

7. PROOF OF THEOREM 1.8

Definition 7.1. Let $f : X \rightarrow X$ be a surjective endomorphism of a normal projective variety X . We say that a positive power of f^* is a *scalar* if there are $s > 0, q > 0$ such that $(f^s)^* H \equiv q^s H$ (numerical equivalence) for any Cartier divisor H , or equivalently, $(f^s)^*|_{N^1(X)} = q^s \text{id}$.

In the example below, f is polarized, but, no positive power of f^* is a scalar.

Example 7.2. Let E be an elliptic curve with no complex multiplication and $Z = E \times E$ an abelian surface with an endomorphism f corresponding to the matrix $\begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}$. Then $\rho(Z) = 3$ and f^* is not a scalar up to a positive power; see [29, Example 4.1.5]. In particular, $f^*|_{N^1(Z)}$ has only one real eigenvalue (counting multiplicities) and the spectral radius of $f^*|_{N^1(Z)}$ is 6; further $f^*H \equiv 6H$ for some nef divisor $H \not\equiv 0$ by applying the Perron-Frobenius theorem to $\text{Nef}(Z)$. We claim that $H^2 > 0$ and hence H is ample; see Remark 4.5. If $H^2 = 0$, then $H^\perp := \{D \in N^1(Z) \mid D \cdot H = 0\}$ is a f^* -invariant 2-dimensional subspace. Note that $H \in H^\perp$. So $f^*|_{N^1(Z)}$ has two real eigenvalues, counting multiplicities. This is a contradiction. Hence f is polarized.

Next we show the existence of a polarized endomorphism $g : S \rightarrow S$, such that no positive power of g^* is a scalar while $(g|_{\text{Alb}(S)})^*$ is a scalar.

Example 7.3. We use the notations in Example 7.2. Let G be a group generated by $\text{diag}[-1, -1]$ and denote by $S = Z/G$ which is a normal K3 surface. Since $q(S) = 0$ and S has rational singularities, the Albanese map is trivial by Lemma 5.1. Note that f is G -equivariant, and $\pi : Z \rightarrow S$ is a finite surjective morphism. So $g = f|_S$ is also polarized by Theorem 3.11. Next we claim that no positive power of g^* is a scalar. Clearly, it suffices to show that $\rho(S) \geq 2$ (then it follows that $\rho(S) = 3$ since $f^*|_{N^1(Z)}$ has only one real eigenvalue, counting multiplicities). Suppose $\rho(S) = 1$. A fibre E_0 of $Z \rightarrow E$ has $E_0^2 = 0$. Since E_0 is G -invariant, $\pi^*\pi(E_0) \equiv aE_0$ for some $a > 0$. Then $0 = a^2 E_0^2 = (\pi^*\pi(E_0))^2 = 4\pi(E_0)^2$ and hence $\pi(E_0)$ is not ample, a contradiction.

Example 7.4. We construct polarized pairs (X, f) such that:

- (1) $\dim(X) = m + n - 1$ with $m \in \{4, 6\}$ and $0 < n < \min\{m, 5\}$,
- (2) X has \mathbb{Q} -factorial terminal (quotient) singularities and is rationally connected,
- (3) the smooth locus X_{reg} of X has infinite algebraic fundamental group $\pi_1^{\text{alg}}(X_{\text{reg}})$,
- (4) $q^{\mathfrak{h}}(X) = n > 0$ (see 2.1 for definition),
- (5) the Iitaka D -dimension satisfies $\kappa(X, -K_X) = m - 1$, and
- (6) the ramification divisor $R_f \subseteq X$ of f is non-trivial.

Indeed, let $G \cong \mathbb{Z}/(m)$ act on \mathbb{P}^{m-1} as a (coordinates) permutation subgroup of S_m so that G has no non-trivial pseudo-reflections (i.e. for any non-trivial $g \in G$, g fixes at most a codimension 2 subset), and \mathbb{P}^{m-1}/G has only canonical singularities. This is guaranteed if the age $a(h) \geq 1$ at every point fixed by a non-trivial h in G , e.g. if $m = 4, 6$; see the proof of [24, Lemma 3]. Let $G = \langle g = \exp(\frac{2\pi\sqrt{-1}}{m}) \rangle$ act diagonally on the abelian variety $A = E^n = E \times \cdots \times E$, with E being an elliptic curve such that G has no non-trivial pseudo-reflection (and hence A/G is Q -abelian) and A/G is

rationally connected. This is achievable by letting $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta_m)$ and choosing suitable $m > n > 0$ (e.g., $m = 4, 6$ and $0 < n < \min\{m, 5\}$); see [7]. Let G act diagonally on $W = \mathbb{P}^{m-1} \times A$. Then $X = W/G$ projects to rationally connected A/G with the general fibre \mathbb{P}^{m-1} and hence it is also rationally connected by [15, Corollary 1.3]. For any non-trivial $g \in G$, $g|_A$ contributes a positive value to the age $a(g)$ and hence $a(g) > 1$. So X is \mathbb{Q} -factorial terminal. Now the multiplication map $\mu_r : A \rightarrow A$, $a \mapsto ra$, with $r \geq 2$, is polarized such that $\mu_r^*H = r^2H$ for any symmetric ample divisor H on A . The power map $q_P : \mathbb{P}^{m-1} \rightarrow \mathbb{P}^{m-1}$, $[X_0 : \cdots : X_{m-1}] \mapsto [X_0^q : \cdots : X_{m-1}^q]$ with $q = r^2$, is also polarized. Thus $f_W = (q_P, \mu_r)$ is a polarized endomorphism of W and it descends to a polarized endomorphism f on X (f_W commutes with the G -action). Since G also has no non-trivial pseudo-reflections on W , the quotient map $\gamma : W \rightarrow X$ is quasi-étale, $K_W = \gamma^*K_X$, and $\kappa(X, -K_X) = \kappa(W, -K_W) = m-1$. Hence the topological fundamental group $\pi_1(X_{\text{reg}})$ of the smooth locus of X is the extension of $\mathbb{Z}/(m)$ by $\mathbb{Z}^{\oplus 2\dim(A)}$, and $q^h(X) = \dim(A) > 0$. For (6), we take D'_i as the pullback to W of the coordinate hyperplane $\{X_i = 0\} \subseteq \mathbb{P}^{m-1}$. Then $f_W^*D'_i = qD'_i$. Now $R_f \geq (q-1)D_i$ with $D_i \subseteq X$ the image of D'_i .

Proof of Theorem 1.8. (1) follows from Lemma 6.9.

If K_X is not pseudo-effective, then by [1, Corollary 1.3.3] we may run MMP with scaling for a finitely many steps: $X = X_1 \dashrightarrow \cdots \dashrightarrow X_j$ (divisorial contractions and flips) and end up with a Mori's Fibre space $X_j \rightarrow X_{j+1}$. Note that X_{j+1} is again \mathbb{Q} -factorial (cf. [21, Corollary 3.18] with klt singularities (cf. [11, Corollary 4.5])). So by running the same program several times, we may get the following sequence:

$$(*) \quad X = X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y,$$

such that K_{X_r} is pseudo-effective. Replacing f by a positive power, polarized f_{i-1} descends to f_i by Lemmas 6.4, 6.5 and Remark 6.3. By Theorem 3.11, f_i is again polarized by some ample Cartier divisor H_i . So the sequence $(*)$ is f -equivariant. Since K_{X_r} is pseudo-effective, $Y = X_r$ is Q -abelian by Lemma 6.9.

By Lemma 5.3, the composition $X_i \dashrightarrow Y$ is a morphism for each i . If $X_i \dashrightarrow X_{i+1}$ is a flip, then for the corresponding flipping contraction $X_i \rightarrow Z_i$, (Z_i, Δ_i) is klt for some effective \mathbb{Q} -divisor Δ_i by [11, Corollary 4.5]. Hence $Z_i \dashrightarrow Y$ is also a morphism by Lemma 5.3 again. Together, the sequence $(*)$ is run over Y .

By Lemmas 2.12 and 5.2, $X_i \rightarrow Y$ is equi-dimensional with every fibre being (irreducible) rationally connected. Since K_X is not pseudo-effective, the final map $X_{r-1} \rightarrow X_r$ is a Fano contraction. So (2) is proved.

Via the pullback, $N^1(X_{i+1})$ can be regarded as a subspace of $N^1(X_i)$ and hence a subspace of $N^1(X)$. Then $f_i^*|_{N^1(X_i)} = f^*|_{N^1(X_i)}$. If $X_i \dashrightarrow X_{i+1}$ is a flip, then $N^1(X_i) =$

$N^1(X_{i+1})$. If $X_i \rightarrow X_{i+1}$ is a divisorial contraction or a Fano contraction, then $N^1(X_{i+1})$ is a codimension one subspace of $N^1(X_i)$ by the cone theorem and $H_i \notin N^1(X_{i+1})$. So $N^1(X_i)$ is spanned by $N^1(X_{i+1})$ and H_i . Together, $N^1(X)$ is spanned by $N^1(Y)$ and those $\{H_i\}_{i < r}$. Clearly, if $i < r$, then $\dim(X_i) > 0$. By Proposition 2.5 and Lemmas 3.1 and 3.10, the eigenvalue of H_i is the same $q = (\deg f_i)^{1/\dim(X_i)}$. So (3) is proved.

(4) is straightforward from (3). \square

8. PROOF OF THEOREM 1.10 AND COROLLARY 1.11

Lemma 8.1. *Let (X, f) be a polarized pair with X a \mathbb{Q} -factorial klt normal projective variety and $q^h(X) = 0$ (which is not automatic even for rationally connected terminal X ; see Example 7.4). Then a positive power of f^* is a scalar.*

Proof. By Theorem 1.8, there is an f^s -equivariant equi-dimensional surjective morphism $\pi : X \rightarrow Y$ with Y a \mathbb{Q} -abelian variety. Let $A \rightarrow Y$ be the finite cover étale in codimension one with A an abelian variety and let X' be the normalization of $X \times_Y A$. Then $p_1 : X' \rightarrow X$ is a finite surjective morphism étale in codimension one. Since $q^h(X) = 0$, $q(A) = 0$ and hence $A = Y$ is a point. Now a positive power of f^* is a scalar by Theorem 1.8. \square

Proposition 8.2. *Let (X, f) be a polarized pair with X a \mathbb{Q} -factorial klt normal projective variety and the irregularity $q(X) = 0$ (this is the case when X is rationally connected). Suppose that $f^*|_{N^1(X)} = q \text{ id}$ for some $q > 1$. Then the following are true.*

- (1) *If the Iitaka D -dimension $\kappa(X, F) = 0$ for a prime divisor F , then $f^{-1}(F) = F$.*
- (2) *Let D_i ($1 \leq i \leq s$) be all the prime divisors with $f^{-1}(D_i) = D_i$ and let $D = \sum_{i=1}^s D_i$. Then $s \leq \dim(X) + \rho(X)$ with $\rho(X)$ the Picard number of X . The equality holds true only when $K_X + D \sim_{\mathbb{Q}} 0$ and hence X is of Calabi-Yau type.*
- (3) *We have the ramification divisor $R_f = (q - 1)D + \Delta$ for some effective divisor Δ , such that Δ and D have no common irreducible component and $\kappa(X, \Delta_j) > 0$ for every irreducible component Δ_j of Δ .*
- (4) *We have $-(K_X + D) \sim_{\mathbb{Q}} \frac{1}{q-1}\Delta$. So either $\Delta \neq 0$ and $\kappa(X, -(K_X + D)) > 0$, or $K_X + D \sim_{\mathbb{Q}} 0$ and hence X is of Calabi-Yau type.*

Proof. We follow the approach of [35, Claim 3.15]. Since $q(X) = 0$, numerical equivalence implies \mathbb{Q} -linear equivalence.

(1) If $f^{-1}(F) \neq F$, then $f^*F \sim_{\mathbb{Q}} qF$ but $f^*F \neq qF$. So $\kappa(X, F) > 0$.

(2) follows from [36, Theorem 1.3].

(3) The ramification index of f along D_i is q for each i , so $R_f = (q - 1)D + \Delta$ for some effective divisor Δ . Clearly, D_i is not an irreducible component of Δ for each i . So, for each j , $f^{-1}(\Delta_j) \neq \Delta_j$ and hence $\kappa(X, \Delta_j) > 0$ by (1).

(4) By the ramification divisor formula, $K_X = f^*K_X + R_f \sim_{\mathbb{Q}} qK_X + (q-1)D + \Delta$. Therefore, $-(K_X + D) \sim_{\mathbb{Q}} \frac{1}{q-1}\Delta$. If $\Delta \neq 0$, then $\kappa(X, -(K_X + D)) > 0$ by (3). If $\Delta = 0$, then $K_X + D \sim_{\mathbb{Q}} 0$. Therefore, (X, D) is log canonical and hence X is of Calabi-Yau type; see [5, Theorem 1.4] or [36, Theorem 1.3]. \square

Lemma 8.3. *Let X be a rationally connected normal projective variety and D a non-uniruled prime divisor such that $K_X + D$ is \mathbb{Q} -Cartier. Then $K_X + D$ is pseudo-effective.*

Proof. Replacing X by a log resolution X' of the pair (X, D) and D by its proper transform D' , we may assume that both X and D are smooth, so that the argument in [27, Theorem 3.7] is applicable. Suppose that $K_X + D$ is not pseudo-effective. Then there is a dominant rational map $X \dashrightarrow Y$ such that D birationally dominates Y . In particular, Y is non-uniruled. This contradicts that X is rationally connected. \square

Proof of Theorem 1.10. Since X is smooth, any quasi-étale morphism onto X is étale by purity of branch loci. Since X is smooth and rationally connected, X has no non-trivial étale cover; see [8, Corollary 4.18]. In particular, $q^h(X) = 0$. Replacing f by a positive power, f^* is a scalar by Lemma 8.1. The assertions (2), (3), and (4) cases (i) and (ii) are straightforward by Proposition 8.2.

For (4) case (iii), by Lemma 8.3, $K_X + D_1$ is pseudo-effective and hence $K_X + D = -\frac{1}{q-1}\Delta$ is pseudo-effective by Proposition 8.2(4). So $\Delta = 0$ and $D = D_1$. Now $K_X + D \sim_{\mathbb{Q}} 0$ and X is of Calabi-Yau type by Proposition 8.2(4). \square

Proof of Corollary 1.11. (1) Since X is not Q -abelian, by Theorem 1.8, K_X is not pseudo-effective. Further, replacing f by a positive power, we may run f -equivariant MMP

$$X = X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y,$$

such that $X_{r-1} \rightarrow X_r$ is a Fano contraction, Y is Q -abelian, and $f|_Y$ is polarized. Since (X, f) is primitive, Y is a point and $X_{i-1} \dashrightarrow X_i$ is either a divisorial contraction or a flip for each $i < r$. Clearly, X_{r-1} is then a Fano variety of Picard number one, so X_{r-1} and hence X are rationally connected; see [17]. By Theorem 1.8(4), f^* is a scalar. Thus Proposition 8.2 is applicable.

(2) By taking pullback, we may regard $N^1(X_i)$ as a subspace of $N^1(X)$ for each i . Let $0 < i_1 < i_2 < \cdots < i_s < r$ (s can be taken as 0) be all the indexes such that $X_i \rightarrow X_{i+1}$ is a divisorial contraction if $i = i_t$ for some $1 \leq t \leq s$. Denote by E_t the exceptional divisor of $X_{i_t} \rightarrow X_{i_t+1}$. Note that $N^1(X_{i_t+1})$ is a codimension 1 subspace of $N^1(X_{i_t})$ by the cone theorem and $-E_t$ is relative ample over X_{i_t+1} by Lemma 2.6. So $N^1(X_{i_t})$ is spanned by $N^1(X_{i_t+1})$ and E_t . Together, $N^1(X)$ is spanned by $N^1(X_{r-1})$ and those E_t (E_t may not be a prime divisor in $N^1(X)$).

Since the MMP is f -equivariant, replacing f by a positive power, we may assume $f^{-1}(E_t(k)) = E_t(k)$ for each irreducible component $E_t(k)$ of E_t . By Proposition 8.2(4), if X is not of Calabi-Yau type, then $K_X + D \sim_{\mathbb{Q}} -\frac{1}{q-1}\Delta \neq 0$. By Proposition 8.2(3), Δ contains no irreducible component like $E_t(k)$. So the image of Δ in X_{r-1} is a non-zero effective divisor. In particular, $N^1(X)$ is spanned by Δ and those E_t , since X_{r-1} is of Picard number 1.

Let A be an ample divisor in $N^1(X)$. Then we may write $A = \sum_{t=1}^s a_t E_t + b\Delta$ for some $a_t, b \in \mathbb{R}$. For the divisor D in Proposition 8.2(2), we may assume $eD = \sum_{t=1}^s E_t + D'$ for some effective divisor D' and $e > 0$. By Proposition 8.2(4), $-cK_X \sim_{\mathbb{Q}} cD + \frac{c}{q-1}\Delta \geq A$ if $c \gg 1$. Therefore, $-K_X$ is big. □

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